Mth232 Exam 3 Name______________

1. If $\mathbf{r}(t)$ is orthogonal to $\frac{d}{dt}\mathbf{r}(t)$ for all t, show that $|\mathbf{r}(t)|$ is constant.

The way here is very simple and straightforward. Orthogonality is characterized by a zero dot-product. Constancy is characterized by a zero derivative. Can you relate those two zeros? Can you find a dot-product equation to differentiate? Yes, the dot product of a unit vector with itself is 1.

$$\frac{d}{dt}|\mathbf{r}(t)|^2 = \frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{r}(t)) = 2\frac{d}{dt}\mathbf{r}(t) \cdot \mathbf{r}(t) = 0$$

Notice that this is an if-and-only-if. However, many people begged the question by assuming that the trajectory is a circle.

2. The osculating plane is the plane spanned by **T** and **N**. Find the equation of the osculating plane of $\mathbf{r} = \sin 2t\mathbf{i} + t\mathbf{j} + \cos 2t\mathbf{k}$ at the point $(0, \pi, 1)$.

The equation of a plane spanned by two vectors (and the vectors in question are T and N evaluated at the point $(0, \pi, 1)$) is given by

$$ax + b(y - \pi) + c(z - 1) = 0,$$

where $\mathrm{T}\times\mathrm{N} = a\mathrm{i}+b\mathrm{j}+c\mathrm{k}$. Notice that you don't actually have to calculate N, since

$$\mathrm{N} = \frac{1}{\kappa}\frac{\partial t}{\partial s}\frac{\partial \mathrm{T}}{\partial t}.$$

In other words, you only need some vector parallel to $\mathrm{T}\times\mathrm{N}$. One such vector is $\mathbf{j} - .5\mathbf{i}$.

3. Suppose that at each point, a curve has the same osculating plane. What is the value of the torsion, τ, along the curve?

Since τ is the rate of change of the unit vector perpendicular to the osculating plane, well, you fill in the blank...

4. Sketch the curve

$$r = \frac{9}{4 - \sqrt{7}\sin\theta},$$

and find the area that it encloses.

The curve is an ellipse, and the area is technically,

$$dA = \frac{1}{2}r^2\,d\theta.$$

However, due to the difficulty of the ensueing integral, that is not the best way to find the area. Much better is to change coordinates. When you do that you'll find that this curve is the same curve as in the next problem (ignoring coordinates)

5. Find the curvature of the curve

$$\frac{x^2}{9} + \frac{y^2}{16} = 1,$$

at the points $(3,0), (0,4)$.

There are several ways of doing this problem. Here is one:

$$\mathbf{r} = 3\cos t\mathbf{i} + 4\sin t\mathbf{j}$$

$$\frac{d}{dt}\mathbf{r} = -3\sin t\mathbf{i} + 4\cos t\mathbf{j}$$

$$\frac{d^2}{dt^2}\mathbf{r} = -3\cos t\mathbf{i} + 4\sin t\mathbf{j}$$

$$|\frac{d}{dt}\mathbf{r} \times \frac{d^2}{dt^2}\mathbf{r}| = |12\sin^2\theta + 12\cos^2\theta| = 12$$

$$\kappa(0) = \frac{3}{16}, \quad \kappa(\frac{\pi}{2}) = \frac{4}{9}$$

Here is another:

$$\text{at} \qquad x = 0, y = 4,$$

$$y = 4 + \frac{dy}{dx}|_{x=0}x + \frac{1}{2}\frac{d^2y}{dx^2}x^2 + o(x^2).$$

Now, differentiate the curve equation twice and get that the curvature is

$$\frac{d^2y}{dx^2} = -\frac{4}{9},$$

since $\frac{dy}{dx} = 0$ at the point. (Other point is done in same way).

I am sure you can find other ways. By the way, notice that for an ellipse, the curvature at the extreme points is

$$\frac{1}{R_1}\frac{R_1^2}{R_2^2}.$$

Can you show this without using calculus?

6. Suppose that a particle moves under the influence of a central force field (where the force depends only on the distance between the particle and the center of the field and is directed along the line joining the particle and the center of the field).

   (a) Show that the angular momentum $\mathbf{r} \times m\mathbf{v}$ is constant.

   $$\frac{d}{dt}(\mathbf{r} \times \frac{d}{dt}\mathbf{r}) = \frac{d}{dt}\mathbf{r} \times \frac{d}{dt}\mathbf{r} + \mathbf{r} \times \frac{d^2}{dt^2}\mathbf{r} = 0 + \mathbf{r} \times (f(r)\mathbf{r}) = 0$$

   (b) Is the motion planar?

   `yes`

   (c) Show that $|\mathbf{r} \times \mathbf{v}| = 2\frac{dA}{dt}$, where A is the area swept out by the position vector, $\mathbf{r}$.
   `infinitesimal area swept out by position vector is the triangle which is half the parallelogram formed by` $\mathbf{r}$ and $d\mathbf{r}$:

   $$dA = \frac{1}{2}|\mathbf{r} \times d\mathbf{r}|$$