Generalizing graphic hyperplane arrangements

Daniel C. Slilaty*

April 15, 2009

Abstract

Let $\mathcal{H} = \{h_1, \dots, h_m\}$ be a central arrangement of hyperplanes in Euclidean n-space $\mathbb{R}^n$. If $x_1, \dots, x_n$ are coordinate variables for $\mathbb{R}^n$, then each h_t is defined by a linear equation $\alpha_{1,t}x_1 + \cdots + \alpha_{n,t}x_n = 0$. The arrangement $\mathcal{H}$ is called *graphic* when each h_t's equation is up to scalar multiplication of the form $x_i - x_j = 0$ for some $i < j$ and $\mathcal{H}$ is called *signed graphic* when each h_t's equation is up to scalar multiplication of the form $x_i = 0$ or $x_i \pm x_j = 0$. In [2] it is shown how the regions of $\mathbb{R}^n$ defined by a graphic arrangement $\mathcal{H}$ correspond with the acyclic orientations of a certain graph defined by $\mathcal{H}$. In [6] it is shown how the regions of $\mathbb{R}^n$ defined by a signed-graphic arrangement $\mathcal{H}$ correspond with the acyclic orientations of a certain signed graph defined by $\mathcal{H}$. Both [2] and [6] use the fact that each hyperplane has a defining equation with at most two nonzero coefficients. Most generally, if a hyperplane is defined by an equation with at most two nonzero coefficients, then that equation is up to scalar multiplication of the form $x_i = 0$ or $x_i - \sigma g x_j = 0$ for some $\sigma \in \{+1, -1\}$ and some real $g > 0$. In this paper we will discuss how such arrangements may be modeled by real gain graphs and we will define orientations and cycles of these gain graphs so that the acyclic orientations are in one-to-one correspondence with the regions in $\mathbb{R}^n$ defined by $\mathcal{H}$.

1 Introduction

Let $\mathcal{H} = \{h_1, \dots, h_m\}$ be an arrangement (i.e., a collection) of hyperplanes in Euclidean n-space $\mathbb{R}^n$ with coordinate variables $x_1, \dots, x_n$. Say that $\mathcal{H}$ is *central*, that is, each $h_t \in \mathcal{H}$ contains the origin. Then each h_t is defined by a linear equation $\alpha_{1,t}x_1 + \cdots + \alpha_{n,t}x_n = 0$. In order to make these representing equations unique, we will assume that each defining equation's first nonzero coefficient is always 1. We will sometimes refer to the hyperplane h_t by this equation that defines it.

The arrangement $\mathcal{H}$ subdivides $\mathbb{R}^n$ into regions where a region R is on one side of each h_t, either the side given by $\alpha_{1,t}x_1 + \cdots + \alpha_{n,t}x_n > 0$ or the side given by $-\alpha_{1,t}x_1 - \cdots - \alpha_{n,t}x_n > 0$. Conversely, let ϵ represent a choice of sides for the hyperplanes in $\mathcal{H}$. We call ϵ an *orientation* of $\mathcal{H}$ and denote $\mathcal{H}$ along with ϵ by $\mathcal{H}_\epsilon$. So now if $A_\mathcal{H}$ is the $n \times m$ matrix $\{\alpha_{i,j}\}$, then ϵ may be represented by the matrix $A_{\mathcal{H}_\epsilon}$ obtained from $A_\mathcal{H}$ by multiplying the column corresponding to h_t by either $+1$ or -1, whichever corresponds to the side of h_t chosen by ϵ. By $-\epsilon$ we mean the orientation that chooses the opposite side of each h_t that ϵ does. We call $-\epsilon$ the *reversal* of ϵ and note that $-A_{\mathcal{H}_\epsilon} = A_{\mathcal{H}_{-\epsilon}}$. Now the orientation ϵ corresponds to a region defined by $\mathcal{H}$ iff $-\epsilon$ corresponds to a region defined by $\mathcal{H}$ iff there is some vector $y \in \mathbb{R}^n$ such that $y^T A_{\mathcal{H}_\epsilon} > 0$. Gordan's Lemma (which is derived from Farkas' Lemma [1]) says that there is a vector $y \in \mathbb{R}^n$ such that $y^T A_{\mathcal{H}_\epsilon} > 0$ iff there is not a nonzero vector $x \in \mathbb{R}^m$ with $x \geq 0$ and $A_{\mathcal{H}_\epsilon} x = 0$. We call a minimal linearly dependent set of columns of $A_\mathcal{H}$ a *circuit* of $\mathcal{H}$, which is a term from matroid theory. If $C \subseteq \mathcal{H}$, then we write the corresponding submatrix of $A_\mathcal{H}$ as A_C and the corresponding submatrix of $A_{\mathcal{H}_\epsilon}$ as A_{C_ϵ}.

Proposition 1.1. *The following are equivalent for an orientation ϵ of $\mathcal{H}$.*

*Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435. Email: daniel.slilaty@wright.edu.

(1) $\mathcal{H}_\epsilon$ corresponds to a region defined by $\mathcal{H}$.

(2) There is no circuit C of $\mathcal{H}$ that has some $x \in \mathbb{R}^k$ with $x > 0$ and $A_{C_\epsilon} x = 0$.

(3) For all circuits C of $\mathcal{H}$ there is y such that $y^T A_{C_\epsilon} > 0$.

Proof. As mentioned in the paragraph preceding this proposition, $\mathcal{H}_\epsilon$ corresponds to a region defined by $\mathcal{H}$ iff there is not a nonzero vector $z \in \mathbb{R}^m$ with $z \geq 0$ and $A_{\mathcal{H}_\epsilon} z = 0$. It is well known that this $z \geq 0$ exists iff there is a minimal linearly dependent set of columns C and nonzero vector $x \geq 0$ such that $A_{C_\epsilon} x = 0$. However since C is minimal we must have that $x > 0$. Farkas' Lemma then says that this vector $x > 0$ exists iff there is no vector y such that $y^T A_{C_\epsilon} > 0$. □

When C and $x > 0$ exist as in Proposition 1.1, C is called a *cycle* of $\mathcal{H}_\epsilon$ (a term from the theory of oriented matroids) and $\mathcal{H}_\epsilon$ is called *acyclic* when it has no cycles. Note that $x > 0$ is unique up to scalar multiplication by the minimality of C.

The arrangement $\mathcal{H}$ is called *graphic* when each $h_t \in \mathcal{H}$ is of the form $x_i - x_j = 0$ for some $i < j$. In the case of a graphic arrangement we can represent $\mathcal{H}$ by the graph $G_\mathcal{H}$ with vertices $x_1, \ldots, x_n$ and a link between x_i and x_j iff $(x_i - x_j = 0) \in \mathcal{H}$. Some terminology for graphs is as follows. If G is a graph, then a *link* is an edge with two distinct endpoints and a *loop* is an edge with both ends attached to the same endpoint. A 2-regular connected subgraph of G is called a *circle* (rather than the more usual term "cycle" which is reserved for the other use mentioned in the previous paragraph). An assignment of one-way directional arrows to the edges of G is called an *orientation* of G. So now when $\mathcal{H}$ is graphic, the matrix $A_\mathcal{H}$ is the usual incidence matrix of the graph $G_\mathcal{H}$ and in the matrix $A_{\mathcal{H}_\epsilon}$ we say that the column corresponding to the expression $x_i - x_j$ represents the edge with a one-way directional arrow on top of it pointing from x_i to x_j and the expression $-x_i + x_j$ has the reverse arrow assigned to the edge. So now the orientation ϵ of $\mathcal{H}$ corresponds to a unique orientation, also call it ϵ, of the graph $G_\mathcal{H}$. Denote this oriented graph (i.e., directed graph) by $G_{\mathcal{H}_\epsilon}$ and now one can check that a cycle of $\mathcal{H}_\epsilon$ corresponds to the edges of a cyclically oriented circle in $G_{\mathcal{H}_\epsilon}$. Thus the acyclic orientations of $G_\mathcal{H}$ correspond to the acyclic orientations of $\mathcal{H}$ which correspond to the regions of $\mathcal{H}$. This correspondence was first discovered by Greene [2].

The arrangement $\mathcal{H}$ is called *signed graphic* when each $h \in \mathcal{H}$ is of the form $x_i = 0$ or $x_i \pm x_j = 0$ for some $i < j$. A *signed graph* is a pair $\Sigma = (G, \sigma)$ where G is a graph and $\sigma : E(G) \to \{+1, -1\}$. When drawing signed graphs we draw positive edges as solid curves and negative edges as dashed curves. One can consider an ordinary graph to be a signed graph in which all edges are signed positively. Given the signed-graphic arrangement $\mathcal{H}$ we associate a signed graph $\Sigma_\mathcal{H}$ on vertices $x_1, \ldots, x_n$ where x_i and x_j are joined by a positive link when $(x_i - x_j = 0) \in \mathcal{H}$ and by a negative link when $(x_i + x_j = 0) \in \mathcal{H}$. There is a negative loop at x_i when $(x_i = 0) \in \mathcal{H}$. In [6], Zaslavsky develops a notion of orientations of a signed graph corresponding to the orientations $\mathcal{H}$. These orientations are defined by giving each edge in $\Sigma_\mathcal{H}$ a *bidirection* which is an assignment of two arrows onto the edge, one at each end. For the column of $A_{\mathcal{H}_\epsilon}$ corresponding to $\pm x_i \pm x_j$ we bidirect the corresponding edge of $\Sigma_\mathcal{H}$ by (for each $k \in \{i, j\}$) assigning an arrow away from vertex x_k at the end for x_k if the coefficient of x_k is $+1$ and an arrow towards vertex x_k at the end for x_k if the coefficient of x_k is -1. Thus a positive edge gets two arrows in the same direction along the edge and a negative edge gets two arrows in opposite directions along the edge. The possibilities for what a bidirected edge can look like are shown in Figure 1. If every arrow given by ϵ at a vertex v is inward or every arrow is outward, then v is called a *singular* vertex for $\Sigma_{\mathcal{H}_\epsilon}$.

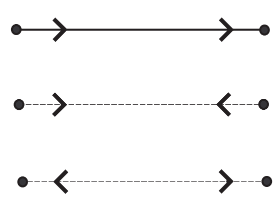


Figure 1.

A circle in a signed graph is considered to be *positive* if the product of signs on its edges is positive. Otherwise, the circle is called *negative*. A circuit of $\mathcal{H}$ corresponds to the edge set of a subgraph of $\Sigma_\mathcal{H}$ that is a positive circle or a subdivision of one of the graphs in Figure 2 that contains only negative circles [4, §5].

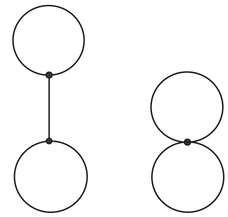


Figure 2.

A cycle for $\Sigma_{\mathcal{H}_\epsilon}$ is then an orientation on one of these circuits with no singular vertex [6]. Up to subdivision and reversal, these cycles are shown in Figure 3 except for cycles that come from positive circles.

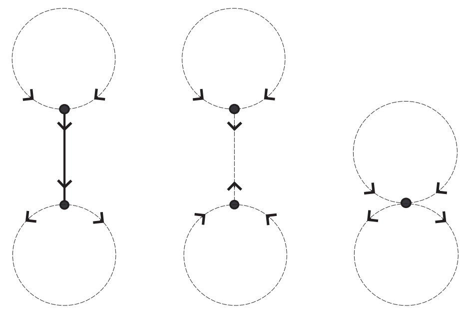


Figure 3.

Both graphic and signed-graphic arrangements take advantage of the fact that the defining equation for each $h_t \in \mathcal{H}$ has at most two nonzero coefficients. In general if $h_t \in \mathcal{H}$ has a defining equation with at most two nonzero coefficients, then we can write such a defining equation as $x_i = 0$ or as $x_i + \sigma_t g_t x_j = 0$ for some $i < j$ where $g_t > 0$ is a real number and $\sigma_t \in \{+1, -1\}$. When each hyperplane in $\mathcal{H}$ has this form we will call $\mathcal{H}$ a *gain-graphic* arrangement. We will represent such an arrangement by a gain graph $\Phi_{\mathcal{H}}$ over the multiplicative group of nonzero real numbers $\mathbb{R}^*$ on vertices $x_1, \ldots, x_n$ where x_i and x_j are joined by a link with gain $\sigma_t g_t$ in the direction from x_i to x_j when $(x_i + \sigma_t g_t x_j = 0) \in \mathcal{H}$. For the hyperplane $x_i = 0$ we have a loop at x_i with gain -1. We will discuss gain graphs in more detail in Section 2. In this paper we develop a notion of orientations and cycles in such gain graphs, describe the structure of the cycles, and show that the acyclic orientations correspond to the regions defined by $\mathcal{H}$. Our main result is Theorem 3.6. The ideas in this paper were the motivation for the more general work done on orienting biased graphs and their matroids in [3].

2 Gain graphs

If e is an edge of G, then we denote e along with a direction along e by $\vec{e}$. The opposite direction along e is then denoted $\vec{e}^{-1}$. This is not the edge e with a one-way directional arrow assigned to it as in directed graphs, but rather a way to specify one direction along an edge from the other. A *walk* $\vec{W}$ in G is a sequence of edges with direction $\vec{e}_1, \ldots, \vec{e}_n$ in which the tip of $\vec{e}_i$ is the tail of $\vec{e}_{i+1}$ for each $i \in \{1, \ldots, n-1\}$. If u is the tail endpoint of $\vec{e}_1$ and v the tip endpoint of $\vec{e}_n$, then $\vec{W}$ is called a uv-walk. If $u = v$, then we call $\vec{W}$ a *closed* walk. By $\vec{W}^{-1}$ we mean the walk $\vec{e}_n^{-1}, \ldots, \vec{e}_1^{-1}$. If a circle C in G has edges $e_1, \ldots, e_n$ in some cyclic ordering around C, then by $\vec{C}$ we mean the closed walk $\vec{e}_1, \ldots, \vec{e}_n$. Of course given C, $\vec{C}$ is not well defined unless we specify a starting edge and direction around C, however, for our purposes this definition will be enough. In figures we will refer to the direction of a walk by placing an arrow alongside it (as in Figure 4) rather than on top of it.

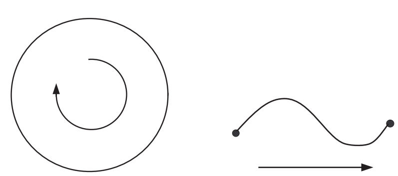


Figure 4.

The set of all possible edges and directions in G is denoted $\vec{E}(G)$. A *gain graph* is a triple $\Phi = (G, \Gamma, \varphi)$ in which G is a graph, Γ a group (which will always be multiplicative in this paper), and $\varphi : \vec{E}(G) \to \Gamma$ a labeling such that $\varphi(\vec{e}^{-1}) = \varphi(\vec{e})^{-1}$. We can extend φ to all walks by setting $\varphi(\vec{W}) = \prod_{i=1}^{n} \varphi(\vec{e}_i)$ and this gives us that $\varphi(\vec{W}^{-1}) = \varphi(\vec{W})^{-1}$. Note that if a circle C satisfies $\varphi(\vec{C}) = 1$ for any one of the possible ways of choosing $\vec{C}$ from C, then, because cyclic shifts of products in groups may be accomplished by conjugation in the group, we get that $\varphi(\vec{C}) = 1 = \varphi(\vec{C}^{-1})$ for all ways of choosing $\vec{C}$ from C. So it is well defined to say that a circle C is *balanced* in Φ when $\varphi(\vec{C}) = 1$ and C is *unbalanced* when $\varphi(\vec{C}) \neq 1$.

3 Real gain graphs and their orientations.

Let $\mathbb{R}^*$ denote the multiplicative group of real numbers. Recall that a gain graphic-arrangement $\mathcal{H}$ has each $h_t \in \mathcal{H}$ of the form $x_i = 0$ or $x_i + \sigma_t g_t x_j = 0$ for $i < j$, $g_t > 0$, and $\sigma_t = \pm 1$. This arrangement is represented by the real gain graph $\Phi_{\mathcal{H}} = (G, \mathbb{R}^*, \varphi)$ on vertices $x_1, \dots, x_n$ where the hyperplane $x_i + \sigma_t g_t x_j = 0$ is represented by a link in $\Phi_{\mathcal{H}}$ between vertices x_i and x_j with gain $\sigma_t g_t$ in the direction from x_i to x_j and the hyperplane $x_i = 0$ represented by a loop at x_i with gain -1. Note now that there is an underlying sign function $\sigma : E(G) \to \{+1, -1\}$ given by $-\sigma_1, -\sigma_2, \dots, -\sigma_m$. We denote this underlying signed graph by $\Sigma_{\Phi_{\mathcal{H}}}$. We say that a circle C is positive (or negative) in $\Phi_{\mathcal{H}}$ when it is positive (or negative) in $\Sigma_{\Phi_{\mathcal{H}}}$. So now if C is balanced in $\Phi_{\mathcal{H}}$ then it is positive in $\Sigma_{\Phi_{\mathcal{H}}}$, if C is negative in $\Sigma_{\Phi_{\mathcal{H}}}$ then it is unbalanced in $\Phi_{\mathcal{H}}$, and the remaining unbalanced cycles in $\Phi_{\mathcal{H}}$ are referred to as *positive unbalanced.* If C is positive unbalanced, then we can choose $\vec{C}$ so that $\varphi(\vec{C}) > 1$ and $0 < \varphi(\vec{C}^{-1}) < 1$. In this case say that $\vec{C}$ is the *preferred* direction along C. A *theta graph* consists of two vertices along with three internally disjoint connecting paths.

Proposition 3.1. *If Θ is a theta subgraph of $\Phi_{\mathcal{H}}$ and all three circles in Θ are positive unbalanced, then there is a planar imbedding of Θ so that the preferred directions of these circles are as in Figure 5.*

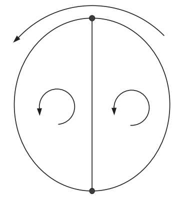


Figure 5.

Proof. Let u and v be the two 3-valent vertices of Θ and let $\alpha_1, \alpha_2, \alpha_3$ be the three uv-paths of Θ. We may assume an imbedding in the plane as in Figure 5 with v at the top and $\alpha_1, \alpha_2, \alpha_3$ from left to right. Now if $\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3$ are uv-walks in Θ corresponding to the three uv-paths, then since Θ has no balanced circles we can assume that $\varphi(\vec{\alpha}_1) > \varphi(\vec{\alpha}_2) > \varphi(\vec{\alpha}_3)$. Evidently then the preferred directions of the three circles of Θ are as in Figure 5. $\square$

Proposition 3.2 (Zaslavsky [5, §2])**.** *If C is a circuit of a gain-graphic arrangement $\mathcal{H}$, then C corresponds to the edge set of a subgraph of $\Phi_{\mathcal{H}}$ that is either a balanced circle, a subdivision of one of the subgraphs in Figure 2 that contains no balanced circles, or a theta subgraph that contains no balanced circles.*

Now consider an orientation ϵ of $\mathcal{H}$. For the column of $A_{\mathcal{H}_\epsilon}$ corresponding to the inequality $\pm x_i + \sigma_t g_t x_j > 0$ with $i < j$ we orient the corresponding edge e of $\Phi_{\mathcal{H}}$ by giving it a bidirection as with signed graphs by assigning an arrow away from x_i at the end for x_i if the coefficient of x_i is $+1$ and an arrow towards x_i at the end for x_i if the coefficient is -1. We assign an arrow away from x_j at the end for x_j if the coefficient of $g_t x_j$ is $+1$ and an arrow towards x_j at the end for x_j if the coefficient is -1.

Now we wish to identify what the cycles are in $\Phi_{\mathcal{H}_\epsilon}$. We start with Propositions 3.3–3.5.

Proposition 3.3. *Let $v_1, e_1, v_2, \dots, v_n, e_n, v_{n+1}$ be the vertices and edges in order along a path P (possibly closed) in $\Phi_{\mathcal{H}}$ and $\vec{P} = \vec{e}_1, \dots, \vec{e}_n$ be the corresponding $v_1 v_{n+1}$ walk. Let ϵ be an orientation of $\Phi_{\mathcal{H}}$ so that no* *vertex aside from v_1 and v_{n+1} in P_ϵ is singular. Then a nonzero solution to $(v_1, \ldots, v_{n+1})A_{P_\epsilon} > 0$ implies that*

$$\tau_1 v_1 + \tau_{n+1}|\varphi(\vec{P})|v_{n+1} > 0$$

in which, for each $j \in \{1, n+1\}$, $\tau_j = +1$ (correspondingly $\tau_j = -1$) if the arrow of P_ϵ at v_j is away from v_j (correspondingly towards v_j).

Proof. If we consider the inequalities in $(v_1, \ldots, v_{n+1})A_{P_\epsilon} > 0$, then because no interior vertex of P_ϵ is singular, we can then scale the inequalities by real positive numbers and add the inequalities together so that all terms for $v_2, \ldots, v_n$ cancel out and yield

$$\tau_1 v_1 + \tau_{n+1}\left|\prod_{i=1}^{n} \varphi(\vec{e}_i)\right| v_{n+1} = \tau_1 v_1 + \tau_{n+1}|\varphi(\vec{P})|v_{n+1} > 0$$

as required. □

Proposition 3.4. *If C is positive unbalanced in $\Phi_{\mathcal{H}}$ and ϵ is an orientation of $\mathcal{H}$ with no singular vertex in C_ϵ, then for each vertex v in C the system of inequalities $y^T A_{C_\epsilon} > 0$ forces*

(1) $v < 0$ if the arrows of C_ϵ at v are in the preferred direction $\vec{C}$ and

(2) $v > 0$ if the arrows of C_ϵ at v are against the preferred direction $\vec{C}$.

Proof. We apply the inequality of Proposition 3.3 where $v_1 = v_{n+1} = v$. If the arrows at v are in the preferred direction of C, then we have

$$v - |\varphi(\vec{P})|v > 0$$

where $|\varphi(\vec{P})| > 1$ and so we get $(1 - |\varphi(\vec{P})|)v > 0$ and so $v < 0$ as required. Our result follows similarly when the arrows at v are against the preferred direction of C. □

Proposition 3.5. *If C is a negative circle in $\Phi_{\mathcal{H}}$ and ϵ is an orientation of $\mathcal{H}$ for which there is only one singular vertex in C_ϵ, say v, then the system of inequalities $y^T A_{C_\epsilon} > 0$ forces*

(1) $v > 0$ if the arrows of C_ϵ at v are both away from v and

(2) $v < 0$ if the arrows of C_ϵ at v are both towards from v.

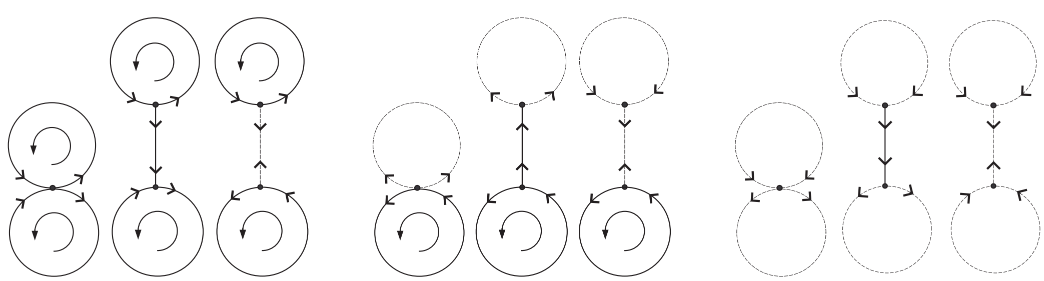
Proof. We apply the inequality of Proposition 3.3 where $v_1 = v_{n+1} = v$. If the arrows at v are both towards v, then Proposition 3.3 yields

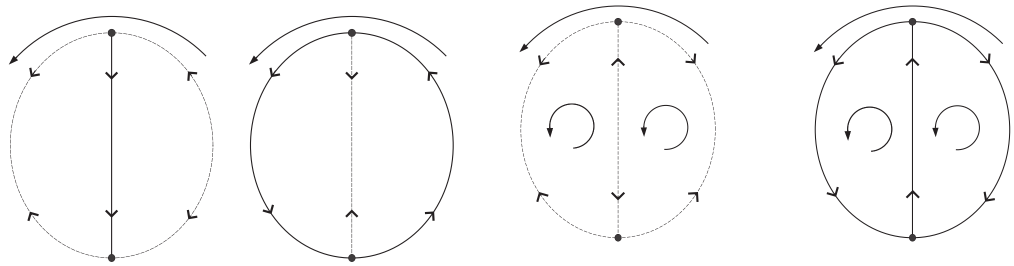
$$-v - |\varphi(\vec{P})|v \geq 0$$

where $|\varphi(\vec{P})| > 1$ and so we get $-(1 + |\varphi(\vec{P})|)v > 0$ and so $v < 0$ as required. Our result follows similarly when the arrows at v are both away from v. □

Now consider the subgraphs of $\Phi_{\mathcal{H}}$ in Figure 6 and ignore the bidirectional arrows for now. A subdivision of one of these subgraphs is one where edges are replaced by paths whose product of signs is the same as that of the replaced edge. So then, aside from balanced circles, these subgraphs represent up to subdivision all the possibilities for sign patterns and preferred directions on circuits of $\Phi_{\mathcal{H}}$ by Propositions 3.1 and 3.2.

Figure 6.





Theorem 3.6 (Main Result)**.** *Let $\mathcal{H}$ be a gain-graphic arrangement. If C_ϵ is a cycle of $\mathcal{H}_\epsilon$, then in $\Phi_{\mathcal{H}_\epsilon}$ the cycle C_ϵ has no singular vertex and is a balanced circle or is a subdivision of a bidirected subgraph from Figure 6 up to reversal.*

Proof. By Proposition 1.1, C_ϵ is a cycle iff there is no solution to the system of inequalities $y^T A_{C_\epsilon} > 0$ iff there is some $x > 0$ such that $A_{C_\epsilon} x = 0$. Furthermore, since C is minimally linearly dependent, the nullspace of A_C has dimension 1 and so there is exactly one cyclic orientation of C up to reversal. Thus we need only identify one orientation of C for which the system of inequalities $y^T A_{C_\epsilon} > 0$ has some inconsistency that prohibits the existence of a solution.

If C_ϵ is a balanced circle with no singular vertex, then Proposition 3.3 gives that for any vertex v in C_ϵ, the inequalities $y^T C_\epsilon > 0$ imply that $v - v > 0$ which is our desired inconsistency and so ϵ and its reversal are the cyclic orientations of balanced circles. Now if C is not a balanced circle, then C minus is bidirectional arrows but along with its sign pattern and preferred directions is shown in Figure 6 up to subdivision. Consider the first circuit in Figure 6. By Proposition 3.4, $y^T C_\epsilon > 0$ implies that the 4-valent vertex is both positive and negative. This is the desired inconsistency and so the orientation shown in Figure 6 along with its reversal are the cyclic orientations. For the second circuit in Figure 2, Proposition 3.4 implies that the upper 3-valent vertex, call it u, is negative in the system of inequalities $y^T C_\epsilon > 0$ and that the lower 3-valent vertex, call it v, is positive in this system. Now Proposition 3.3 gives us the inequality $u - gv > 0$ for some $g > 0$. This is the desired inconsistency and so the orientation shown in Figure 6 along with its reversal are the cyclic orientations. The remainder of the cases are handled similarly. □

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